Quiz 5 Solution

Tuesday, September 24, 2013 3:50 PM

Evaluate the capacity values for the channels whose transition probabilities are given by each of the following matrices Q.

(a) $Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

(b) $Q = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}$

(c) $Q = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 2/3 & 1/6 & 1/6 \end{bmatrix}$

(d) $Q = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(a) The channel is symmetric. So,

$$C = \log_2 |S_Y| - H(\underline{r}) = \log_2 2 - H\left(\left[\tfrac{1}{3}\ \tfrac{2}{3}\right]\right)$$

$$\approx 1 - 0.9183 \approx 0.0817.$$

(b) The channel is symmetric. So

$$C = \log_2 |S_Y| - H(\underline{r}) = \log_2 3 - H\left(\left[\tfrac{1}{6}\ \tfrac{1}{3}\ \tfrac{1}{2}\right]\right)$$

$$\approx 0.1258.$$

(c) Note that the rows of Q are all the same.

So, $H(Y|x) = H\left(\left[2/3\ \ 1/6\ \ 1/6\right]\right)$ for any x.

Therefore, $H(Y|X) = \sum_x p(x) H(Y|x) = H\left(\left[\tfrac{2}{3}\ \tfrac{1}{6}\ \tfrac{1}{6}\right]\right) \approx 1.2516$

Next, note that for any $p(x)$, we have

$$q(y) = \sum_x p(x) \underbrace{Q(y|x)} = \underbrace{Q(y|x)}.$$

For a given y, this is the same for all x.

So, $H(Y) = H(Y|x)$ as well.

$I(X;Y) = H(Y) - H(Y|X) = 0$ for any $\underline{p}$.

Remark. We can also conclude from $q(y) = Q(y|x)$ that $X \perp\!\!\!\perp Y$.
This also implies $I(X;Y) = 0$.

$$C = \max_{\underline{p}} I(X;Y) = 0.$$

(d) Note that this is a noisy channel with non overlapping outputs.

So, $C = \log_2 |S_X| = \log_2 3 \approx 1.5850$.